

Capital budgeting problems with fuzzy cash flows*

Christer Carlsson

christer.carlsson@abo.fi

Robert Fullér

<http://www.abo.fi/~rfuller/robert.html>

Abstract

We consider the *internal rate of return* (IRR) decision rule in capital budgeting problems with fuzzy cash flows. The possibility distribution of the IRR at any $r \geq 0$, is defined to be the degree of possibility that the (fuzzy) net present value of the project with discount factor r equals to zero. Generalizing our earlier results on fuzzy capital budgeting problems [5] we show that the possibility distribution of the IRR is a highly nonlinear function which is getting more and more unbalanced by increasing imprecision in the future cash flow. However, it is stable under small changes in the membership functions of fuzzy numbers representing the linguistic values of future cash flows.

Keywords: Capital budgeting problem, internal rate of return, possibility distribution, sensitivity analysis

1 Introduction

Many decision making problems concern projects in which the cost and benefits accrue over a number of years. In this paper we consider only cases in which the costs and benefits are entirely monetary, such as the capital budgeting or capital investment decisions arising in commerce and industry. Authors consider two kinds of decision problems in capital budgeting: *accept-or-reject* and *ranking*. In accept-or-reject decisions, each project is considered independently of all other projects. Thus a portfolio of accepted projects is built up from several independent decisions. In ranking decisions, all the available projects are compared and ranked in order of favourability with the intention of adopting a single project: the most favourable. It should be noted that it is often important to include a null project representing the status quo; all the projects may be unfavourable compared with the alternative of adopting none of them (if this is possible). Several decision rules have been suggested [1, 6, 9] to help decision makers rank projects which involve timestreams of costs and benefits, such as the *payback period*, *accounting rate of return* (ARR), *internal rate of return* (IRR) and *net present value* (NPV).

We shall briefly describe just the IRR decision rule. Let $\{a_0, a_1, \dots, a_n\}$ be a given net cash flow of a project a over n periods. We assume that $a_0 < 0$ as the project starts with an initial investment. The IRR, denoted by r^{**} , is defined to be the value of r such that the NPV of the project is zero. Thus find the IRR of a we need to solve

$$S(a, r) := a_0 + \frac{a_1}{1+r} + \dots + \frac{a_n}{(1+r)^n} = 0 \quad (1)$$

It is well-known that, if there is reinvestment in a project ($a_i < 0$ for some $i \geq 1$) then its IRR may become ill-defined, i.e. equation (1) may have more than one solution. If the IRR of a project is ill-defined, it is not a suitable criterion to use in either accept-or-reject or ranking decisions. Suppose,

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however, that no project considered involves any reinvestment. Then NPV is a strictly monotone decreasing function of r and the equation (1) has a unique solution, moreover, the discount rate r can be interpreted in strictly financial terms as an interest rate. Now in an accept-or-reject decision it is clear that, if the market rate of interest is r_0 , the project should be accepted if $r^{**} > r_0$ because this implies that NPV at r_0 is positive. In comparing two projects, the one with the higher IRR should be preferred.

2 IRR with fuzzy cash flows

More often than not future cash flows (and interest rates) are not known exactly, and we have to work with their estimations, such as 'around 5,000 in the next few years' (or 'close to 3 %'). Fuzzy numbers appear to be an adequate tool to represent imprecisely given cash flows [3, 4, 7, 13, 14].

Definition 2.1 A fuzzy number A is a fuzzy set of the real line with a normal, (fuzzy) convex and continuous membership function of bounded support. The family of fuzzy numbers will be denoted by \mathcal{F} .

A fuzzy set A is called a symmetric triangular fuzzy number with center a and width $\alpha > 0$ if its membership function has the following form

$$A(t) = \begin{cases} 1 - \frac{|a - t|}{\alpha} & \text{if } |a - t| \leq \alpha \\ 0 & \text{otherwise} \end{cases}$$

and we use the notation $A = (a, \alpha)$. If $\alpha = 0$ then A collapses to the characteristic function of $\{a\} \subset \mathbb{R}$ and we write $A = \bar{a}$.

We will use symmetric triangular fuzzy numbers to represent the values of the linguistic variable [16] *cash*.

If $A = (a, \alpha)$ and $B = (b, \beta)$ are fuzzy numbers of symmetric triangular form and $\lambda \in \mathbb{R}$ then $A + B$, $A - B$ and λA are defined by the extension principle in the usual way:

$$A + B = (a + b, \alpha + \beta), \quad A - B = (a - b, \alpha + \beta), \quad \lambda A = (\lambda a, |\lambda| \alpha).$$

Furthermore, if $A_i = (a_i, \alpha_i)$ and $\lambda_i = 1/(1 + r)^i$, $i = 0, 1, \dots, n$, then we get

$$A_0 + \sum_{i=1}^n \frac{A_i}{(1 + r)^i} = \left(a_0 + \frac{a_1}{1 + r} + \dots + \frac{a_n}{(1 + r)^n}, \alpha_0 + \frac{\alpha_1}{1 + r} + \dots + \frac{\alpha_n}{(1 + r)^n} \right). \quad (2)$$

Let A and $B \in \mathcal{F}$ be fuzzy numbers. The degree of possibility that the proposition "A is equal to B" is true denoted by $\text{Pos}[A = B]$ and defined by the extension principle as

$$\text{Pos}[A = B] = \sup_{x \in \mathbb{R}} \min\{A(x), B(x)\} = (A - B)(0), \quad (3)$$

The Hausdorff distance of A and B , denoted by $D(A, B)$, is defined by [12]

$$D(A, B) = \max_{\theta \in [0, 1]} \max\{|a_1(\theta) - b_1(\theta)|, |a_2(\theta) - b_2(\theta)|\}$$

where $[a_1(\theta), a_2(\theta)]$ and $[b_1(\theta), b_2(\theta)]$ denote the θ -level sets of A and B , respectively. For example, if $A = (a, \alpha)$ and $B = (b, \alpha)$ are fuzzy numbers of symmetric triangular form with the same width $\alpha > 0$ then

$$D(A, B) = |a - b|.$$

Lemma 2.1 [10] Let $\delta > 0$ be a real number, and let $A = (a, \alpha)$ and $B = (b, \beta)$ be symmetric triangular fuzzy numbers. Then from the inequality $D(A, B) \leq \delta$ it follows that

$$\sup_{t \in \mathbb{R}} |A(t) - B(t)| \leq \max \left\{ \frac{\delta}{\alpha}, \frac{\delta}{\beta} \right\}. \quad (4)$$

Let $\{A_0 = (a_0, \alpha_0), A_1 = (a_1, \alpha_1), \dots, A_n = (a_n, \alpha_n)\}$ be a given net fuzzy cash flow of a project A over n periods. By replacing the crisp cash flow values with fuzzy numbers in (1) we get

$$A_0 + \frac{A_1}{1+r} + \dots + \frac{A_n}{(1+r)^n} = \bar{0} \quad (5)$$

where the equation is defined in possibilistic sense, and $\bar{0}$ denotes the characteristic function of zero. That is, the fuzzy solution [2] of (5) is computed by

$$\mu_{IRR}(r) = \text{Pos} \left[A_0 + \sum_{i=1}^n \frac{A_i}{(1+r)^i} = \bar{0} \right] = \left(A_0 + \sum_{i=1}^n \frac{A_i}{(1+r)^i} \right)(0).$$

for each $r \geq 0$. Using the definition of possibility (3) and representation (2) we find

$$\mu_{IRR}(r) = \begin{cases} 1 - \frac{|S(a, r)|}{S(\alpha, r)} & \text{if } |S(a, r)| \leq S(\alpha, r), \\ 0 & \text{otherwise} \end{cases}$$

where we used the notations

$$S(a, r) = a_0 + \frac{a_1}{1+r} + \dots + \frac{a_n}{(1+r)^n}, \quad S(\alpha, r) = \alpha_0 + \frac{\alpha_1}{1+r} + \dots + \frac{\alpha_n}{(1+r)^n}$$

We assume that $a_0 < 0$ (the project starts with an initial investment), $a_0 \leq a_1 + \dots + a_n$ (the project is at least as good as the null project), and $a_i \geq 0$, $i = 1, \dots, n$, (no reinvestment). In this case we always get quasi-triangular fuzzy numbers for IRR in \mathbb{R}_0^+ and equation (5) has a unique maximizing solution, r^* , such that,

$$\mu_{IRR}(r^*) = \max_{r \geq 0} \mu_{IRR}(r) = 1,$$

and r^* coincides with r^{**} , which is the internal rate of return of the (crisp) project $a = (a_0, a_1, \dots, a_n)$. Really, if $r \geq 0$ then $\mu_{IRR}(r) = 1$ if and only if $S(a, r) = 0$.

As an example consider a 4-year project

$$A = \{(-5, \alpha), (3, \alpha), (4, \alpha), (6, \alpha), (10, \alpha)\},$$

with fuzzy IRR,

$$\mu_{IRR}(r) = \begin{cases} 1 - \frac{\left| -5 + \frac{3}{1+r} + \frac{4}{(1+r)^2} + \frac{6}{(1+r)^3} + \frac{10}{(1+r)^4} \right|}{\alpha \left[1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \frac{1}{(1+r)^4} \right]} & \text{if } |S(a, r)| \leq S(\alpha, r), \\ 0 & \text{otherwise} \end{cases}$$

It is easy to compute that $\mu_{IRR}(0.781) = 1$ for all $\alpha \geq 0$, so the maximizing solution to possibilistic equation (5) is independent of α .

However, the possibility distribution of the IRR is getting more and more unbalanced as the widths of the fuzzy numbers are growing. This means that when comparing the fuzzy IRR with the market

interest rate r_0 in an accept-or-reject decision, the defuzzified value of μ_{IRR} will definitely differ from r^* whenever the process of defuzzification takes into account all points with positive membership degrees (and not only the maximizing point).

For example, all projects in Figs.1-3, have the same maximizing solution $r^* = 0.781$, but if we employ the center-of-gravity method then the defuzzified value of the project with $\alpha_0 = \alpha_1 = \dots = \alpha_n = 5$ is around 0.84, which is essentially bigger (in terms of rates of return) than 0.781.

In ranking decisions we have to compare possibility distributions of a non-symmetric quasi-triangular form.

3 Sensitivity analysis in fuzzy capital budgeting

Consider two projects $A = \{A_0, A_1, \dots, A_n\}$ and $A^\delta = \{A_0^\delta, A_1^\delta, \dots, A_n^\delta\}$ with fuzzy cash flows $A_i = (a_i, \alpha_i)$ and $A_i^\delta = (a_i^\delta, \alpha_i)$, $i = 0, 1, \dots, n$. The fuzzy IRR of project A^δ , denoted by μ_{IRR}^δ , is computed by

$$\mu_{IRR}^\delta(r) = \text{Pos} \left[A_0^\delta + \sum_{i=1}^n \frac{A_i^\delta}{(1+r)^i} = \bar{0} \right] = \left(A_0^\delta + \sum_{i=1}^n \frac{A_i^\delta}{(1+r)^i} \right)(0).$$

for each $r \geq 0$. Using the definition of possibility (3) and representation (2) we find

$$\mu_{IRR}^\delta(r) = \begin{cases} 1 - \frac{|S(a^\delta, r)|}{S(\alpha, r)} & \text{if } |S(a^\delta, r)| \leq S(\alpha, r), \\ 0 & \text{otherwise} \end{cases}$$

where we used the notation

$$S(a^\delta, r) = a_0^\delta + \frac{a_1^\delta}{1+r} + \dots + \frac{a_n^\delta}{(1+r)^n}.$$

Let $r^{**}(\delta)$ denote the IRR of the crisp project $a^\delta = (a_0^\delta, a_1^\delta, \dots, a_n^\delta)$. That is,

$$S(a^\delta, r^{**}(\delta)) = a_0^\delta + \frac{a_1^\delta}{1+r^{**}(\delta)} + \dots + \frac{a_n^\delta}{(1+r^{**}(\delta))^n} = 0 \quad (6)$$

In the following we suppose that $r^{**}(\delta)$ is the only solution to equation (6), i.e. $a_0^\delta < 0$ and $a_i^\delta > 0$ for $i = 1, \dots, n$.

The next theorem shows that if the centers of fuzzy numbers A_i and A_i^δ in projects A and A^δ are close to each others, then there can only be a small deviation in the possibility distributions of their fuzzy IRR.

Theorem 3.1 *Let $\delta > 0$ be a real number. If*

$$\max\{|a_0 - a_0^\delta|, |a_1 - a_1^\delta|, \dots, |a_n - a_n^\delta|\} \leq \delta$$

then

$$\max_{r \geq 0} |\mu_{IRR}(r) - \mu_{IRR}^\delta(r)| \leq \min \left\{ 1, \frac{\delta}{\alpha_{\max}} \right\}. \quad (7)$$

where

$$\alpha_{\max} = \max\{\alpha_0, \alpha_1, \dots, \alpha_n\}$$

μ_{IRR} and μ_{IRR}^δ are the possibility distributions of IRR of projects A and A^δ , respectively.

Proof. It is sufficient to show that

$$\begin{aligned}
& |\mu_{IRR}(r) - \mu_{IRR}^\delta(r)| = \\
& \left| \text{Pos} \left[A_0 + \sum_{i=1}^n \frac{A_i}{(1+r)^i} = \bar{0} \right] - \text{Pos} \left[A_0^\delta + \sum_{i=1}^n \frac{A_i^\delta}{(1+r)^i} = \bar{0} \right] \right| = \\
& \left| \left(A_0 + \sum_{i=1}^n \frac{A_i}{(1+r)^i} \right)(0) - \left(A_0^\delta + \sum_{i=1}^n \frac{A_i^\delta}{(1+r)^i} \right)(0) \right| \leq \min \left\{ 1, \frac{\delta}{\alpha} \right\} \quad (8)
\end{aligned}$$

for any $r \geq 0$, because (7) follows from (8). Using representation (2) and applying Lemma 2.1 to

$$A_0 + \sum_{i=1}^n \frac{A_i}{(1+r)^i} = (S(a, r), S(\alpha, r)),$$

and

$$A_0^\delta + \sum_{i=1}^n \frac{A_i^\delta}{(1+r)^i} = (S(a^\delta, r), S(\alpha, r)),$$

we find

$$\begin{aligned}
& D \left(A_0 + \sum_{i=1}^n \frac{A_i}{(1+r)^i}, A_0^\delta + \sum_{i=1}^n \frac{A_i^\delta}{(1+r)^i} \right) = |S(a, r) - S(a^\delta, r)| = \\
& \left| a_0 + \frac{a_1}{1+r} + \cdots + \frac{a_n}{(1+r)^n} - \left(a_0^\delta + \frac{a_1^\delta}{1+r} + \cdots + \frac{a_n^\delta}{(1+r)^n} \right) \right| \leq \\
& |a_0 - a_0^\delta| + \frac{1}{1+r} \times |a_1 - a_1^\delta| + \frac{1}{(1+r)^n} \times |a_n - a_n^\delta| \leq (n+1) \times \delta,
\end{aligned}$$

for any $r \geq 0$, and

$$\begin{aligned}
& \left| \left(A_0 + \sum_{i=1}^n \frac{A_i}{(1+r)^i} \right)(0) - \left(A_0^\delta + \sum_{i=1}^n \frac{A_i^\delta}{(1+r)^i} \right)(0) \right| \leq \\
& \sup_{t \in \mathbb{R}} \left| \left(A_0 + \sum_{i=1}^n \frac{A_i}{(1+r)^i} \right)(t) - \left(A_0^\delta + \sum_{i=1}^n \frac{A_i^\delta}{(1+r)^i} \right)(t) \right| \leq \\
& \max \left\{ \frac{(n+1)\delta}{\alpha_0 + \alpha_1 + \cdots + \alpha_n} \right\} \leq \max \left\{ \frac{(n+1)\delta}{(n+1) \max\{\alpha_0, \alpha_1, \dots, \alpha_n\}} \right\} = \left\{ \frac{\delta}{\alpha_{\max}} \right\}.
\end{aligned}$$

Which ends the proof.

Theorem 3.1 can also be extended to fuzzy cash flows with arbitrary (continuous) fuzzy numbers.

Theorem 3.2 *Let $\delta > 0$ be a real number. If*

$$\max\{D(A_0, A_0^\delta), D(A_1, A_1^\delta), \dots, D(A_n, A_n^\delta)\} \leq \delta$$

then

$$\max_{r \geq 0} |\mu_{IRR}(r) - \mu_{IRR}^\delta(r)| \leq \min\{1, \omega(\delta)\}.$$

where $\omega(\delta)$ denotes the maximum of moduli of continuity of all the fuzzy numbers in projects A and A^δ at point δ .

The proof of this theorem is carried out analogously to the proof of Theorem 3.1 in [8].

4 Concluding remarks

In this paper we have shown that the fuzzy IRR has a stability property under small changes in the membership functions representing the fuzzy cash flows. Nevertheless, the behavior of the maximizing solution, $r^*(\delta)$, of possibilistic equation

$$A_0^\delta + \sum_{i=1}^n \frac{A_i^\delta}{(1+r)^i} = \bar{0},$$

towards small perturbations in the membership functions of the fuzzy coefficients can be very fortuitous. That is, the distance

$$|r^* - r^*(\delta)|,$$

(which coincides with $|r^{**} - r^{**}(\delta)|$, the distance between the internal rates of returns of crisp projects $a = (a_0, a_1, \dots, a_n)$ and $a^\delta = (a_0^\delta, a_1^\delta, \dots, a_n^\delta)$ if $A_i = (a_i, \alpha_i)$ and $A_i^\delta = (a_i^\delta, \alpha_i)$, $i = 0, 1, \dots, n$) can be very big even for very small δ .

In this manner, the fuzzy model can be considered as a well-posed extension [11, 15] of the (generally) ill-posed crisp internal rate of return decision rule.

If the fuzzy numbers in projects A and A^δ are not strictly unimodal (for example trapezoidal) then the set of maximizing solutions of the fuzzy IRR is a segment of the real line. In this case any IRR obtained from a crisp project, in which the future cash values are chosen from the cores of the corresponding fuzzy numbers, belongs to the core of the fuzzy IRR.

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References

- [1] J. Baron, *Thinking and Deciding*, 2nd edition, (Cambridge University Press, 1994).
- [2] R.E. Bellman and L.A. Zadeh, Decision-making in a fuzzy environment, *Management Sciences*, Ser. B 17(1970) 141-164.
- [3] J.J. Buckley, Portfolio analysis using possibility distributions, in: E. Sanchez and L.A. Zadeh eds., *Approximate Reasoning in Intelligent Systems, Decision and Control*, Pergamon Press, New-York, 1987 69-76.
- [4] J.J. Buckley, Fuzzy mathematics of finance, *Fuzzy Sets and Systems*, 21(1987) 257-273.
- [5] C. Carlsson and R. Fullér, On fuzzy capital budgeting problem, in: Proceedings of the International ICSC Symposium on Soft Computing in Financial Markets, Rochester, New York, USA, June 22-25, 1999, ICSC Academic Press, 1999 (to appear).
- [6] Robert T. Clemen, *Making Hard Decisions: An Introduction to Decision Analysis*, 2nd edition, (Duxbury Press, An Imprint of Wadsworth Publishing Company, Belmont, California, 1996).
- [7] C. Chiu and C.S. Park, Fuzzy cash flow analysis using present worth criterion, *The Engineering Economist*, 39(1994) 113-138.
- [8] M.Fedrizzi and R. Fullér, Stability in possibilistic linear programming problems with continuous fuzzy number parameters, *Fuzzy Sets and Systems*, 47(1992) 187-191.

- [9] Simon French, *Readings in Decision Analysis*, (Chapman and Hall, London, 1990).
- [10] R.Fullér, On stability in possibilistic linear equality systems with Lipschitzian fuzzy numbers, *Fuzzy Sets and Systems*, 34(1990) 347-353.
- [11] R. Fullér, Well-posed fuzzy extensions of ill-posed linear equality systems, *Fuzzy Systems and Mathematics*, 5(1991) 43-48.
- [12] R.Goetschel and W.Voxman, Topological properties of fuzzy numbers, *Fuzzy Sets and Systems*, 10(1983), 87-99.
- [13] E.E. Haven, The use of fuzzy multicriteria analysis in evaluating projects with identical net present values and risk, in: M. Mares et al eds., *Proceedings of Seventh IFSA World Congress*, June 25-29, 1997, Prague, Academia Prague, Vol. III, 1997 154-157.
- [14] C. Kahraman and Z. Ulukan, Fuzzy cash flows under inflation, in: M. Mares et al eds., *Proceedings of Seventh IFSA World Congress*, June 25-29, 1997, Prague, Academia Prague, Vol. IV, 1997 104-108.
- [15] M. Kovács, A stable embedding of ill-posed linear systems into fuzzy systems, *Fuzzy Sets and Systems*, 45(1992) 305–312.
- [16] L.A.Zadeh, The concept of linguistic variable and its applications to approximate reasoning, Parts I,II,III, *Information Sciences*, 8(1975) 199-251; 8(1975) 301-357; 9(1975) 43-80.